Hyper-extensionality and one-node elimination on membership graphs

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Abstract. A (hereditarily finite) set/hyperset S can be completely depicted by a (finite pointed) graph $\mathcal{G}_S$—dubbed its *membership graph*—in which every node represents an element of the *transitive closure* of $\{S\}$ and every arc represents a membership relation holding between its source and its target. In a membership graph different nodes must have different sets of successors and, more generally, if the graph is *cyclic* no *bisimilar* nodes are admitted. We call such graphs *hyper-extensional*.
Therefore, the elimination of even a single node in a membership graph can cause different nodes to "collapse" (becoming representatives of the same set/hyperset) and the graph to loose hyper-extensionality and its original membership character.
In this note we discuss the following problem: given S is it always possible to find a node s in $\mathcal{G}_S$ whose deletion does not cause *any* collapse?
Keywords: Hereditarily Finite Sets, Hypersets, Bisimulation, Membership Graphs.

Introduction

Two sets are equal if and only if they have the same elements. This principle—the so-called axiom of *Extensionality*—goes at the very heart of the notion of set, as it states that given s and s', the condition of them having the same elements is sufficient to guarantee that s and s' are the *same thing*. As a matter of fact, extensionality not only was among the postulates of the first axiomatisation of Set Theory—i.e. the Zermelo-Fraenkel axiomatic set theory ZF—but is also undisputedly present in any subsequent axiomatic presentation of sets.

Being able to establish equality by extensionality only, however, presupposes that membership is *acyclic*. In fact, admitting the possibility to have a cyclic membership relation, imagine two objects a and b satisfying the following simple set-theoretic equation $x = \{x\}$. In this case, in order to establish wether a is equal to b using extensionality, we must rely on our ability of establishing equality between their elements. That is equality between ... a and b. Our argument (as the underlying membership relation) becomes cyclic!

Since the 1980s, the elegant notion of BISIMILARITY has been extensively used to sensibly extend the notion of set-theoretic equality to the case in which we drop the assumption that the membership relation must be acyclic. The notion of bisimilarity was introduced (almost at the same time) in many different fields. Aczel, in particular, set up a graph-theoretic view on sets and hypersets, according to which the consequences of dropping acyclicity of $\in$ was rendered cleanly in its *anti-foundation axiom* (*AFA* [Acz88], see also [BM96]), stated in terms of bisimilarity.

In this note we study a simple-looking problem that can be stated on the graph-theoretic representation of sets and hypersets. The problem can be, informally, given as follows: given a set S and its membership graph G_S—a graph representing the transitive closure of S—, does there always exist a node in G_S (i.e. a set in the transitive closure of S) whose elimination from G_S will cause no pair of nodes to become bisimilar? In other words, is it always possible to find a way to *reduce* a graph-theoretic representation of a hyperset by one element, without losing *any* inequality among the remaining hypersets in the transitive closure of S?

Notice that we pose and study the question in the hereditarily finite case. That is, not only we play with pure sets (i.e. sets whose only elements are themselves sets), but also on an entirely finite "chessboard".

The question has an easy and positive answer for well-founded sets using the notion of rank. However, as the guidance for choosing which node to delete is exactly the feature we cannot count on when dealing with hypersets (that is the notion of *rank*) the case in which $\in$ can be cyclic becomes quickly more interesting.

We present here a few partial and initial results that, incidentally, suggest that probably the problem should be studied as a graph-theoretic one.

In the concluding remarks we briefly discuss a problem that, among others, brought us to get interested in the above mentioned question.

1 Basics

Below we schematically recall some basic definitions. See [Jec78] and [Lev79] for detailed definitions. For a given well-founded set x we say that x is *hereditarily finite* if it is finite and all its elements are hereditarily finite as well. In formulae: $\mathsf{HF}(x) \ \Leftrightarrow_{\text{Def}} \mathsf{Is\_finite}(x) \wedge \forall y \in x \ \mathsf{HF}(y)$. Moreover, we define the *rank* and the *transitive closure* of x as follows[4] : $\mathsf{rk}(x) =_{\text{Def}} \sup\{\, \mathsf{rk}(y)+1 : y \in x \,\}$, with $\mathsf{rk}(\emptyset) = 0$, and $\mathsf{trCl}(x) =_{\text{Def}} x \cup \bigcup\{\, \mathsf{trCl}(y) : y \in x \,\}$.

If, as we do here, we do not assume $\in$ to be necessarily well-founded, a few words are in order to reasonably extend the notion of hereditarily finite set and of transitive closure. In fact, also the notion of rank can be redesigned for the non-well-founded arena[5]. In order to state the anti-foundation axiom and capture

[4] These definitions can be fully formally given by induction on $\in$, by exploiting any sensible notion of *finiteness*.

[5] Actually, this can be done in many different ways, but the real power of any such extension remains rather mysterious (see [PP04,DPP04]).

more clearly the notion of hyperset, we need to specify the above mentioned extension of the principle of extensionality.

To introduce hereditarily finite hypersets we need the definition of *bisimulation relation*. This definition is first given for graphs—as follows—and then is used as an equality criterion to introduce the world of hypersets. This last step is done exploiting the fact that both sets and hypersets are naturally understood as *membership* graphs.

Definition 1. *A* BISIMULATION *on* (V, E) *is a relation* $\flat \subseteq V \times V$ *that satisfies*

1) *to every child* v_0 *of* u_0 *there corresponds at least one child* v_1 *of* u_1 *such that* $v_0 \flat v_1$ *holds, and*
2) *to every child* v_1 *of* u_1 *there corresponds at least one child* v_0 *of* u_0 *such that* $v_0 \flat v_1$ *holds.*

At this point we can define BISIMILARITY to be the relation $\equiv_{(V,E)}$ (or simply $\equiv$) defined between nodes $u, v \in V$ as: $u \equiv_{(V,E)} v$ iff $u \flat v$ holds for *some* bisimulation $\flat$ on (V, E). It plainly turns out that $\equiv_{(V,E)}$ is a bisimulation (actually, the largest of all bisimulations) on (V, E); moreover, it is an equivalence relation over V. The following definitions (given following [Acz88]) establish the bridge between graphs and sets.

Definition 2. *A* POINTED *graph* $\mathcal{G} = (G, v)$ *is a graph* $G = (V, E)$ *with a distinguished node* $v \in V$ *(its point) from which every node in* V *is* E*-reachable.*

Definition 3. *Given a set* S*, its* MEMBERSHIP GRAPH $\mathcal{G}_S$ *is the pointed graph* (G_S, S)*, where* $G_S = (\mathsf{trCl}(\{S\}), E_S)$ *with*

$$E_S = \{\langle v, w\rangle : v \in \mathsf{trCl}(\{S\}) \wedge w \in \mathsf{trCl}(\{S\}) \wedge w \in v\}$$

With a slight abuse of terminology we will say that graph G (not pointed) is a membership graph if there exists a node s in the graph G such that (G, s) is isomorphic to a membership graph. An acyclic membership graph corresponds to the transitive closure of a well-founded set. Below we give two simple results implying that bisimulation is, in fact, coherent with the extensionality principle.

Proposition 1. *The membership graph of any hereditarily finite set has the identity relation as its only bisimulation.*

Any finite, acyclic, pointed graph having identity as its only bisimulation is isomorphic to the membership graph of a hereditarily finite set.

On the basis of the above proposition, one can identify HF (i.e. the collection of x's such that $\mathsf{HF}(x)$) with the collection of those finite, acyclic, pointed graphs whose only bisimulation is the identity —which, in turn, is the collection of those finite acyclic pointed graphs in which no two different nodes have the same successor set. We can now proceed to define *hypersets* simply by dropping the acyclicity requirement and using bisimulation as equality criterion.

Definition 4. *A* HYPERSET *is (the isomorphism class of) a pointed graph on which identity is the only bisimulation. Such an entity is said to be* HEREDITARILY FINITE *when it has finitely many nodes.*

Recalling that the SUBGRAPH ISSUING FROM w in a graph G is the subgraph, pointed in w, that consists of all nodes which are reachable from w in G, we can readily introduce the membership relation between hypersets as follows.

Definition 5. *Given two hypersets h and $h' = (G, v)$, with $G = (V, E)$ as usual, we say that $h \in h'$ if h is (isomorphic to) the pointed subgraph of G issuing from a node w with $\langle v, w \rangle \in E$.*

The class of hereditarily finite hypersets includes the class of hereditarily finite sets. From now on we will identify any hypersets S (possibly well-founded) with its membership graph $\mathcal{G}_S$—that is, with a representative of its isomorphism class. Moreover, we will say that a graph (not necessarily a membership graph) is *hyper-extensional* if its only bisimulation is the identity.

2 One-element elimination

Consider a hyperset S and recall that, by definition, $\mathcal{G}_S$ is hyper-extensional. For any given $s \in \mathsf{trCl}(\{S\})$, we denote by $G_S - s$ the graph obtained from G_S by eliminating s together with all the arcs incident to s. Notice that it is possible that $G_S - s$ is *not* a membership graph (e.g., the case in which $s = S$). As we said in the introduction, the question we want to discuss in this note is whether, given a hyperset S, it is always possible to find $s \in \mathsf{trCl}(\{S\})$ such that $G_S - s$ is hyper-extensional. Clearly, if G_S is acyclic the question has a positive answer, as $G_S - S$ is undoubtedly hyper-extensional. However, at least in the well-founded case, it is always possible to maintain (hyper-)extensionality even eliminating a node $s \in \mathsf{trCl}(S)$ in such a way that $G_S - s$ remains a membership graph.

Proposition 2. *Given a hereditarily finite set S there exists an $s \in \mathsf{trCl}(S)$ such that $(G_S - s, S)$ is (isomorphic to) a membership graph.*

Proof. (Sketch) We can determine s as follows: if there exist two elements of the same rank in the transitive closure of S, let r be the maximum such rank and take s to be any element in the transitive closure of S of rank r. Otherwise take s as the empty set.

The general case in which $\mathcal{G}_S$ is cyclic is more challenging. First of all, we observe that we can produce a scenario in which the only possible eliminable s is in fact the point S.

Example 1. Consider the hyperset satisfying the following system of set-theoretic equations: $S = \{T\}$, $T = \{U, S\}$, $U = \{T, \emptyset\}$. In the above case the only eliminable element in $\mathcal{G}_S$ is its point S.

The above example marks a difference between the well-founded and the non well-founded case, as it tells us that the generalisation of Proposition 2 to the cyclic case does not hold. However, it leaves the question open as whether, possibly by permitting the elimination of the point, it is always possible to delete a node from G_S having the remaining graph hyper-extensional.

Definition 6. *Let G_S^{scc} be the graph having scc's of G_S as nodes, and an arc between A and B if and only if there exist an arc in G_S having source in A and target in B.*

Proposition 3. *For any membership graph $\mathcal{G}_S$, the graph G_S^{scc} is acyclic and has at most two sinks.*

Even though—as we said—it is not easy to chose a notion of rank for non well-founded sets, let the rank of A of G_S to be the length of the longest path in G_S^{scc} from A to a sink. We do not know if, given a membership graph $\mathcal{G}_S$, a node whose elimination does not disrupt hyper-extensionality always exists. However, if this is the case, one such node must always be found in the strongly connected component of maximal rank.

Proposition 4. *If any membership graph $\mathcal{G}_S$ admits an $s \in \mathsf{trCl}(\{S\})$ such that $G_S - s$ is hyper-extensional, then there exists such an s in the strongly connected component of G_S of maximal rank.*

Proof. (Sketch) By contradiction, let $\mathcal{C}$ an scc of maximum rank that does not contain any eliminable node. Represent $\mathcal{C}$ (that must be unique) as a labelled graph, where labels correspond to hypersets constituting the part of the transitive closure built on strongly connected components of ranks smaller than that of $\mathcal{C}$. At this point prove that a collection $\mathcal{L}$ of sets exists, such that: (a) the elements of $\mathcal{L}$ can be used in place of the original labels to discriminate (sets of) nodes in $\mathcal{C}$; (b) no element in $\mathcal{L}$ can be eliminated without causing a collapse, either among nodes in $\mathcal{L}$ or in $\mathcal{C}$. The graph obtained from G_S maintaining $\mathcal{C}$ and using $\mathcal{L}$ in place of the original labels, does not admit the elimination of *any* node, contradicting the hypothesis.

On the one hand, a reasonable point of view could be that a choice for an eliminable node should be strictly tied with a definition of some notion of rank compatible with cyclic structures. On the other hand, one could argue that on cyclic graphs an eliminable node must be characterised by *two* different features: a maximal rank—captured by the maximality of the strongly connected component where the node *must* be chosen—, and a different—unknown—feature, related with the cyclic character of the graph and guiding in the choice within the strongly connected component.

Concluding remarks

We consider that the problem presented here is simple and elegant enough to deserve a (computationally well characterised) answer without any further consideration. However, let us conclude by mentioning a context in which the question tackled here was raised, and for which a (positive) answer would be beneficial.

In [PT13] the problem of generating uniformly and at random a set with a given number of elements in its transitive closure was studied. The proposed solution was based on generating extensional acyclic digraphs with a given number of labeled vertices (since all of the $n!$ labelings of the vertices of an extensional acyclic digraph, or of a hyper-extensional digraph on n vertices, lead to non-isomorphic labelled digraphs).

The results in [PT13] are based on a Markov chain Monte Carlo-based algorithm, initially proposed for generating acyclic digraphs [MDBM01,MP04]. The key fact needed in order to show that the Markov chain converges to the uniform distribution were the irreducibility, aperiodicity, and symmetry of the chain. The idea exploited in the construction of the Markov chain was to show that a pair of elementary operations on graphs (implemented as basic transition rules of the Markov chain, akin to the elimination of a node) could be used to transform any graph G into another graph G' within the same family.

Even though this problem was later solved in [RT13] by a deterministic algorithm based on a combinatorial decomposition (and a resulting counting recurrence), as mentioned above, we are far from having such a counter-part for hyper-extensional digraphs. However, a positive answer to the question posed in this note would allow one to extend the Markov chain Monte Carlo technique to the realm of hypersets, which would be the first result of its kind.

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